PARTIAL DIFFERENTIAL EQUATIONS

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2. Laplace equation and eigenfunctions

(1) Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume f is a continuous function such that $\int_\Omega fw = 0$ for all $w \in C_c^\infty(\Omega)$. Prove that it must be $f \equiv 0$ in Ω.

(2 points)

(2) Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume $f \in L^1(\Omega)$ is such that $\int_\Omega fw = 0$ for all $w \in C_c^\infty(\Omega)$. Prove that it must be $f \equiv 0$ in Ω almost everywhere.

Hint: Use Lebesgue Differentiation Theorem.

(3 points)

(3) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Prove the Mean Value Property: if $u \in C^2(\overline{\Omega})$ satisfies $\Delta u = 0$ in Ω, then

$$u(x) = \fint_{B_r(x)} u \quad \text{for all} \quad B_r(x) \subset \Omega.$$

Deduce the maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: Show that $\fint_{B_r(x)} u = \fint_{B_1} u(x + rz)dz$, and differentiate in r to show that this integral is constant in r.

(3 points)

(4) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that $v \in C^2(\overline{\Omega})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in} \quad \Omega.$$

(i) Similarly to the previous exercise, prove that for any subharmonic function v we have

$$v(x) \leq \fint_{B_r(x)} v \quad \text{for all} \quad B_r(x) \subset \Omega.$$

(ii) Prove that $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$. Is the same true for the minimum of v?

(iii) Let $\phi \in C^\infty(\mathbb{R})$ be a convex function, and $u \in C^2$ be harmonic in $\Omega \subset \mathbb{R}^n$. Prove that the function $\phi(u)$ is subharmonic.

(iv) Prove that if u is harmonic in $\Omega \subset \mathbb{R}^n$, then $v := |\nabla u|^2$ is subharmonic.

(4 points)

(5) Let $\Omega \subset \mathbb{R}^n$ be any bounded smooth domain, and let $u \in C^2(\overline{\Omega})$. Show that the following are equivalent:

(i) u is a solution of the Dirichlet problem

$$\begin{cases} \Delta u &= f & \text{in} \quad \Omega \\ u &= 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(ii) u minimizes the energy functional

$$\mathcal{E}(w) = \int_\Omega \tfrac{1}{2}|\nabla w|^2 + \int_\Omega fw$$

among all functions $w \in C^2(\overline{\Omega})$ satisfying $w = 0$ on $\partial\Omega$.

(3 points)

(6) Let $\Omega \subset \mathbb{R}^n$ be any bounded smooth domain, and let $u \in C^2(\overline{\Omega})$. Show that the following are equivalent:

(i) u is a solution of the Neumann problem

$$\begin{cases} \Delta u &= f & \text{in} \quad \Omega \\ \partial_\nu u &= 0 & \text{on} \quad \partial\Omega. \end{cases}$$

where ∂_ν is the normal derivative.

(ii) u minimizes the energy functional

$$\mathcal{E}(w) = \int_\Omega \tfrac{1}{2}|\nabla w|^2 + \int_\Omega fw$$

among all functions $w \in C^2(\overline{\Omega})$.

(4 points)

(7) Let $\Omega \subset \mathbb{R}^n$ be any bounded smooth and connected domain. Show that the only solutions $u \in C^2(\overline{\Omega})$ of the Neumann problem

$$\begin{cases} \Delta u &= 0 & \text{in} \quad \Omega \\ \partial_\nu u &= 0 & \text{on} \quad \partial\Omega \end{cases}$$

are $u \equiv c$, for some constant c.

(2 points)

(8) Let $\Omega \subset \mathbb{R}^n$ be any bounded smooth domain, and let $f \in L^2(\Omega)$ be such that $\int_\Omega f = 0$. Prove that there exists a minimizer $u \in H^1(\Omega)$ of the functional

$$\mathcal{E}(w) = \int_\Omega \tfrac{1}{2}|\nabla w|^2 + \int_\Omega fw$$

among all functions $w \in H^1(\Omega)$.

What happens if $\int_\Omega f \neq 0$?

Hint: Notice that $\mathcal{E}(w) = \mathcal{E}(w + c)$ for any constant $c \in \mathbb{R}$. To prove the existence of a minimizer, take functions with $\int_\Omega u = 0$, and use the Poincaré inequality from Exercise (12) in Chapter 1.

(4 points)

(9) If we denote by $\lambda_k(\Omega)$ the k-th eigenvalue corresponding to the domain $\Omega \subset \mathbb{R}^n$, then what are the eigenvalues of the rescaled domain

$$r\Omega = \{rx \in \mathbb{R}^n : x \in \Omega\},$$

for $r > 0$?

(2 points)

(10) Let $Q = (0, \pi) \times (0, \pi)$ be a square in $\mathbb{R}^2$.

(i) Find, by separation of variables, all eigenfunctions of the form $\phi(x_1, x_2) = \xi(x_1)\eta(x_2)$, as well as their corresponding eigenvalues λ.

(ii) Show that these eigenfunctions form a basis for $L^2(Q)$, and deduce that there are no more eigenfunctions other than the ones you found.

(iii) Denote as usual $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k$ the ordered eigenvalues for the Laplacian in Q (counted with multiplicity). Prove that

$$\lambda_k \asymp k \quad \text{as} \quad k \to \infty.$$

(4 points)

(11) (i) Let $Q = (0, \pi)^n$ be a hypercube in $\mathbb{R}^n$. Follow the steps of the previous exercise to show that for such domain Q we have

$$\lambda_k \asymp k^{2/n} \quad \text{as} \quad k \to \infty.$$

(ii) Combine this with the conclusion of the next exercise to show that, for *any* bounded domain $\Omega \subset \mathbb{R}^n$, we have

$$\lambda_k \asymp k^{2/n} \quad \text{as} \quad k \to \infty.$$

(4 points)

(12) (i) Let $0 < \lambda_1 < \lambda_2 \leq ...$ be the eigenvalues of the Laplacian in Ω with Dirichlet boundary conditions. Prove that

$$\lambda_k = \max_{f_1,...,f_{k-1} \in L^2(\Omega)} \min_{\substack{u \in H_0^1(\Omega) \\ \int_\Omega u f_i = 0 \\ \|u\|_{L^2(\Omega)} = 1}} \|\nabla u\|_{L^2(\Omega)},$$

where the maximum is taken among all possible choices of $f_1, ..., f_{k-1} \in L^2(\Omega)$.

(ii) Deduce that, if we denote $\lambda_k(\Omega)$ the k-th eigenvalue for the domain Ω, then

$$\Omega' \subset \Omega \qquad \Longrightarrow \qquad \lambda_k(\Omega') \geq \lambda_k(\Omega)$$

for all $k = 1, 2, ...$

(4 points)

(13) Let $\Omega \subset \mathbb{R}^n$ be any bounded smooth and connected domain. Prove that there exists a sequence $0 = \mu_1 < \mu_2 \leq ... \leq \mu_n \to \infty$ of eigenvalues for the Laplacian with Neumann boundary conditions, with eigenfunctions $\varphi_k \in H^1(\Omega)$ solving

$$\begin{cases} -\Delta\varphi_k = \mu_k \varphi_k & \text{in} \quad \Omega \\ \partial_\nu \varphi_k = 0 & \text{on} \quad \partial\Omega \end{cases}$$

in the weak sense.

Prove also that these eigenfunctions φ_k form an orthonormal basis for $L^2(\Omega)$.

(4 points)